

Word values in p -adic and adelic groups

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March 6, 2013

1 Introduction

In recent years there has been wide interest in word maps on groups, with various connections and applications to other areas; see for instance [Bo], [LiSh], [La], [Ja], [Sh1], [Sh2], [Sh3], [LSH1], [LSH2], [LSH3], [GSh], [LBST], [NP], [Se2], [ScSh], [LST1], [LST2], as well as Segal's book [Se] and the references therein.

Recall that a word $w = w(x_1, \dots, x_d)$ is an element of a free group F_d on the free generators x_1, \dots, x_d . Given such a word w and a group G we define a word map from G^d to G induced by substitution. The image of this map is denoted by $w(G)$.

The study of this image attracted considerable attention. By Borel [Bo] (see also [La]) word maps corresponding to a non-trivial word $w \neq 1$ are dominant on simple algebraic groups. Word maps on finite simple groups were subsequently studied extensively. A major result from [LST1] shows that, given a word $w \neq 1$, and a sufficiently large finite simple group G , every element of G is a product of two values of w , namely $w(G)^2 = G$.

For some special words even more has been shown. If $w = [x_1, x_2]$, the commutator word, then [LBST] shows that $w(G) = G$ for all finite simple groups, proving a longstanding conjecture of Ore [O] (see also [EG] and the references therein).

In this paper we study similar problems for certain infinite groups, namely p -adic groups, and, more generally, simple algebraic groups over local rings. This involves new challenges and new methods. In [Sh3] it is conjectured that every element of $\mathrm{SL}_n(\mathbb{Z}_p)$ (p prime, $n \geq 2$) is a commutator (assuming $p > 3$ if $n = 2$), and some preliminary results were obtained. In Theorem 3.8 below we prove this conjecture for $\mathrm{PSL}_n(\mathbb{Z}_p)$ assuming that n is a proper

divisor of $p - 1$. Moreover, considering the very general setup $G = \mathrm{SL}_n(O)$ where O is a local ring whose residue field O/m has more than $n + 1$ elements, we show that every matrix which is not congruent to a scalar modulo m can be expressed as a commutator (see Theorem 3.5).

We also study arbitrary word maps in p -adic groups. A result of Jaikin-Zapirain [Ja] shows that every word has finite width in a compact finitely generated p -adic analytic group. This means that every element of the verbal subgroup generated by the image of the word map is a bounded product of word values and their inverses. Finding explicit bounds on the word width is in general a very challenging problem.

Consider the p -adic group $G(\mathbb{Z}_p)$, where G is a semisimple, simply connected, algebraic group over \mathbb{Q} . For any fixed word $w \neq 1$ and large p we show that $w(G(\mathbb{Z}_p))^3 = G(\mathbb{Z}_p)$, so in particular the word width is at most 3 (see Theorem 2.3 below, which is somewhat stronger). It turns out that this result cannot be improved to $w(G(\mathbb{Z}_p))^2 = G(\mathbb{Z}_p)$, since by [LST2] finite quasisimple groups H need not satisfy $w(H)^2 = H$ (when $w \neq 1$ is fixed and H is as large as we like). However, we show in Theorem 2.4 below that every element of $G(\mathbb{Z}_p)$ whose image modulo the first congruence subgroup $G^1(\mathbb{Z}_p)$ is not central is a product of two values of w . We also prove some results on word maps on adelic groups.

Acknowledgement. We thank Michael Larsen for useful comments. Avni was supported by NSF Grant DMS-0901638, Gelander was supported by ERC grant 203418 and ISF grant 1003/11, Kassabov was supported by NSF grant DMS-0900932, and Shalev was supported by advanced ERC grant 247034.

2 General Words

In this section, we lift some results about the possible values of word maps from algebraic groups over finite fields to algebraic groups over local rings of characteristic 0. To simplify notation, assume that $O = \mathbb{Z}_p$. If G is an algebraic group over \mathbb{Z}_p , we denote the kernel of $G(\mathbb{Z}_p) \rightarrow G(\mathbb{F}_p)$ by $G^1(\mathbb{Z}_p)$. For any element $w(x_1, \dots, x_d) = x_{i_1}^{m_1} x_{i_2}^{m_2} \cdots x_{i_k}^{m_k} \in F_d$, we define a map, which we also denote by w , from G^d to G by $w(g_1, \dots, g_d) = g_{i_1}^{m_1} g_{i_2}^{m_2} \cdots g_{i_k}^{m_k}$. Since $G(\mathbb{Z}_p)$ and $G^1(\mathbb{Z}_p)$ are p -adic analytic and word maps w are analytic, the derivative (or differential) dw is well defined (in terms of local coordinates it is given by the Jacobian matrix).

Lemma 2.1. *Let G be a semisimple algebraic group over \mathbb{Q} . For every non-trivial word w , $w(G(\mathbb{Z}_p))$ contains an open subset of $G(\mathbb{Z}_p)$.*

Proof. A theorem of Borel [Bo] (see also [La]) says that the map $w : G^d \rightarrow G$ is dominant. Over an algebraically closed field of characteristic 0, like $\overline{\mathbb{Q}_p}$, this is equivalent to the existence of a point for which the derivative of w is surjective (as a map of $\overline{\mathbb{Q}_p}$ vector spaces). Since the set of points in G^n for which dw is surjective is Zariski open and $G(\mathbb{Z}_p)$ is Zariski dense, there is a tuple $\vec{g} = (g_1, \dots, g_d) \in G(\mathbb{Z}_p)^d$ such that $dw|_{\vec{g}}$ is surjective. The lemma now follows from the p -adic version of the open mapping theorem (cf. [Ser, Part II, Ch. 3, Sec. 9]). \square

Remark 2.2. *This lemma does not hold for local rings of positive characteristic: for example, the image of $\mathrm{SL}_n(\mathbb{F}_q[[t]])$ under the map $x \mapsto x^p$ is contained in the set of all matrices all of whose eigenvalues are p -th powers, which is a nowhere dense set.*

Theorem 2.3. *Suppose that G is a semisimple, simply connected, algebraic group over \mathbb{Q} , and that w_1, w_2, w_3 are non-trivial words. If p is large enough, then $w_1(G(\mathbb{Z}_p)) \cdot w_2(G(\mathbb{Z}_p)) \cdot w_3(G(\mathbb{Z}_p)) = G(\mathbb{Z}_p)$. In particular $w(G(\mathbb{Z}_p))^3 = G(\mathbb{Z}_p)$ for $w \neq 1$ and sufficiently large p .*

Proof. Let $k = \mathbb{F}_p$. If p is large enough, G_k is simply connected, $\dim G_k = \dim G$, and $G_k(k)$ is isomorphic to the quotient of $G(\mathbb{Z}_p)$ by its first congruence subgroup. Let n_i be the number of letters appearing in w_i , $i = 1, 2, 3$. Since w_1 is dominant, $\dim w_1^{-1}(g) < n_1 \dim(G)$ for every $g \in G_k$. Since the varieties $w_1^{-1}(g)$ have bounded degree, we get that there is a constant c such that $|w_1^{-1}(g)| < c|k|^{n_1 \dim(G)-1}$ for every $g \in G_k(k)$, and that the set of points $x \in G(k)^{n_1}$ for which the derivative of w_1 at x is surjective has size greater than $\frac{1}{c}|k|^{n_1 \dim(G)}$.

Fix $g \in G(\mathbb{Z}_p)$. From the previous paragraph, if p is large, there is $x \in G(\mathbb{Z}_p)^{n_1}$ such that $w_1(x)$ is not congruent to any element in $gZ(G)$ modulo p , and the derivative of w_1 at x is surjective. By Hensel's Lemma, the last condition implies that $w_1(G(\mathbb{Z}_p))$ contains the $G^1(\mathbb{Z}_p)$ -coset of $w_1(x)$. By [LSH1, Theorem 3.3], there are $y \in G(\mathbb{Z}_p)^{n_2}$ and $z \in G(\mathbb{Z}_p)^{n_3}$ such that $w_2(y)w_3(z)$ is congruent modulo p to $w_1(x)^{-1}g$. But then, $w_1(G(\mathbb{Z}_p)) \cdot w_2(G(\mathbb{Z}_p)) \cdot w_3(G(\mathbb{Z}_p))$ contains the whole $G^1(\mathbb{Z}_p)$ -coset of $w_1(x)w_2(y)w_3(z)$. In particular, it contains g . \square

Note that the condition on p is necessary, since for small p w_i may be an identity on some finite image of $G(\mathbb{Z}_p)$.

The story for product of two word values is more complicated. Indeed, even for finite quasisimple groups, not every element is a product of two word values. However, the exceptional elements must be in the center.

Theorem 2.4. *Let w_1, w_2 be non-trivial words, and let G be a semi-simple simply connected algebraic group over \mathbb{Q} . If p is large enough, then $w_1(G(\mathbb{Z}_p)) \cdot w_2(G(\mathbb{Z}_p)) \supset G(\mathbb{Z}_p) \setminus (Z(G) \cdot G^1(\mathbb{Z}_p))$.*

To prove the theorem we need some preparations. The following Lemma is proved in [EG]:

Lemma 2.5. *Suppose G is a simply connected Chevalley group over a finite field k of size greater than 4. Let C_1, C_2 be two conjugacy classes of regular elements from the maximally split torus in $G(k)$. Then $C_1 \cdot C_2 \supset G(k) \setminus Z(G)$.*

Lemma 2.6. *Suppose that $f : G \rightarrow H$ is a homomorphism, X_1, X_2 are subsets of H and $X_1 \cdot X_2 = F \subset H$. Then $f^{-1}(X_1) \cdot f^{-1}(X_2) = f^{-1}(F)$.*

Proof. Let $g \in f^{-1}(F)$. By assumption, there are $g_1 \in f^{-1}(X_1)$ and $g_2 \in f^{-1}(X_2)$ such that $f(g) = f(g_1)f(g_2)$. Therefore, $g_1^{-1}g \in g_2 \ker(f) \subset f^{-1}(X_2)$. Hence, there is $g_3 \in f^{-1}(X_2)$ such that $g = g_1g_3$. \square

We will make use of the following:

Lemma 2.7. *Let $V \subset G^d$ be a proper rational sub-variety. For p large enough, there is a d -tuple $(x_1, \dots, x_d) \in G^d(k) \setminus V(k)$ such that $w_1(x_1, \dots, x_d)$ is a regular element in a split torus.*

Proof. Since $w : G^d \rightarrow G$ is dominant and its differential is surjective on a Zariski open set, the restriction $w|_{G^d \setminus V}$ is also dominant. Let $W \subset G$ be a proper algebraic subset containing $G \setminus w(G \setminus V)$.

By the Lang–Weil estimates (see, for example [LW]), $|W(k)| \leq Cp^{\dim G - 1}$ for some constant C . On the other hand, by [LST1], there is a constant c such that at least $c|G(k)|$ elements that are regular in some split torus are w_1 -values. Hence, there is a word value that is both a regular element in a split torus, and not in $W(k)$. \square

Proof of Theorem 2.4. Let $k = \mathbb{F}_p$. If p is large enough then G_k is semisimple, and the map $G(\mathbb{Z}_p) \rightarrow G(k)$ is onto. Moreover, in this case, Borel's theorem implies that the map $w_1 : G^n \rightarrow G$ is dominant, and, hence, generically smooth; let $V \subset G^n$ be the locus of points where dw_1 is not surjective.

Pick $(g_1, \dots, g_d) \in G(\mathbb{Z}_p)^d \setminus V(\mathbb{Z}_p)$. The derivative of w_1 at (g_i) is a \mathbb{Z}_p -linear map between the relative tangent space to G^d at (g_i) over $\text{Spec } \mathbb{Z}_p$, which is identified with $\mathfrak{g}(\mathbb{Z}_p)^d$, and the relative tangent space to G at $w_1(g_i)$, which is identified with $\mathfrak{g}(\mathbb{Z}_p)$. Since the reduction of this map is onto, it follows that the map itself is onto. It follows that the whole coset $w_1(g_i)G^1(\mathbb{Z}_p)$ is contained in the image of $G(\mathbb{Z}_p)^d$ under w_1 . Since $w_1(G(\mathbb{Z}_p))$ is closed under conjugation, it follows that it contains the pre-image of a regular semi-simple conjugacy class. The same is true for the image of w_2 . By Lemmas 2.5 and 2.6, it follows that $w_1(G(\mathbb{Z}_p)) \cdot w_2(G(\mathbb{Z}_p))$ contains all elements whose reduction modulo p is not central. \square

We now draw some conclusions for adelic groups. Results on finite word width in some adelic groups were proved by Segal in [Se2].

Corollary 2.8. *Let G be a semi-simple connected algebraic group over \mathbb{Q} .*

(i) *For any two non-trivial words, w_1 and w_2 , the set $w_1(G(\widehat{\mathbb{Z}}))w_2(G(\widehat{\mathbb{Z}}))$ has positive measure in $G(\widehat{\mathbb{Z}})$.*

(ii) *If w_3 is another non-trivial word, then $w_1(G(\widehat{\mathbb{Z}})) \cdot w_2(G(\widehat{\mathbb{Z}})) \cdot w_3(G(\widehat{\mathbb{Z}}))$ is open in $G(\widehat{\mathbb{Z}})$.*

Proof. First we note that, for any p and any non-trivial word w , the set $w(G(\mathbb{Z}_p))$ has a non-empty interior. Indeed, since w is dominant, there is $(g_1, \dots, g_d) \in G(\mathbb{Z}_p)$ such that the derivative of w at (g_1, \dots, g_d) is surjective.

Suppose that Theorem 2.4 holds for $p > p_0$.

The set $K = \prod_{p \leq p_0} w_1(\mathbb{Z}_p)w_2(\mathbb{Z}_p)$ is open, so has positive measure λ_K in $\prod_{p \leq p_0} G(\mathbb{Z}_p)$. Since

$$w_1(\widehat{\mathbb{Z}})w_2(\widehat{\mathbb{Z}}) = \prod_p w_1(\mathbb{Z}_p)w_2(\mathbb{Z}_p) \supset K \times \prod_{p > p_0} (G(\mathbb{Z}_p) \setminus (Z(G) \cdot G^1(\mathbb{Z}_p))),$$

we get that the measure of $w_1(\widehat{\mathbb{Z}})w_2(\widehat{\mathbb{Z}})$ is greater than or equal to

$$\lambda_K \cdot \prod_p \left(1 - \frac{|Z(G)|}{|G(\mathbb{F}_p)|}\right) \geq \lambda_K \cdot \prod_p \left(1 - \frac{C}{p^{\dim G}}\right)$$

for some constant C . Since the dimension of G is greater than 1, the product converges to a positive number.

The second claim follows similarly from Theorem 2.3.

□

3 The Commutator Word

In this section we consider the image of the commutator map in special linear groups over local rings of arbitrary characteristic. Let O be a local ring with residue field K and maximal ideal \mathfrak{m} . We set $G = \mathrm{SL}_n$, and let $\mathfrak{g}, \mathfrak{g}^*$ be the Lie algebra of G and its dual. The Killing form $\langle X, Y \rangle = \mathrm{trace}(XY)$ is conjugation-invariant and non-degenerate for every K .

Let $G^k(O)$ denote the k -th congruence subgroup of $G(O)$, i.e., the kernel of $G(O) \rightarrow G(O/\mathfrak{m}^k)$. It is well known that $G^k(O)/G^{k+1}(O)$ is isomorphic (as a $G(O)$ -module) to $\mathfrak{g}(K) \otimes_K \mathfrak{m}^k/\mathfrak{m}^{k+1}$, and the action of $G(O)$ on the tensor product is via the action of $G(K)$ on $\mathfrak{g}(K)$.

Proposition 3.1. *Let $\bar{g}_1, \bar{g}_2 \in G(K)$ be elements such that the group $H = \langle \bar{g}_1, \bar{g}_2 \rangle \subset G(K)$ does not have any fixed vectors in $\mathfrak{g}^*(K)$. Then for any lift g of $[\bar{g}_1, \bar{g}_2]$ to $G(O)$ there are lifts g_1 and g_2 of \bar{g}_1 and \bar{g}_2 such that $g = [g_1, g_2]$.*

Proof. It is enough to show that, under the assumptions of the proposition, the derivative of the commutator map at (\bar{g}_1, \bar{g}_2) is onto. We first compute the derivative of the commutator map. Suppose that $g_1, g_2 \in G$, that $X, Y \in \mathfrak{g}$, and that $\epsilon^2 = 0$. Then

$$\begin{aligned} [g_1(1 + \epsilon X), g_2(1 + \epsilon Y)] &= g_1(1 + \epsilon X)g_2(1 + \epsilon Y)(1 - \epsilon X)g_1^{-1}(1 - \epsilon Y)g_2^{-1} = \\ &= [g_1, g_2] + \epsilon (g_1 X g_2 g_1^{-1} g_2^{-1} + g_1 g_2 Y g_1^{-1} g_2^{-1} - g_1 g_2 X g_1^{-1} g_2^{-1} - g_1 g_2 g_1^{-1} Y g_2^{-1}) = \\ &= [g_1, g_2] \left(1 + \epsilon \left(X^{g_2 g_1^{-1} g_2^{-1}} + Y^{g_1^{-1} g_2^{-1}} - X^{g_1^{-1} g_2^{-1}} - Y^{g_2^{-1}} \right) \right) \end{aligned}$$

So the derivative of the commutator map at (g_1, g_2) is

$$[(X, Y) \mapsto (X^{g_2} - X)^{g_1^{-1}} + (Y^{g_1^{-1}} - Y)]^{g_2^{-1}}$$

For any $g \in G$, the image of $Z \mapsto Z^g - Z$ is the orthogonal complement, with respect to the Killing form $\langle \cdot, \cdot \rangle$, of the centralizer of g because

$$(\forall Z) \langle Z^g - Z, W \rangle = 0 \iff (\forall Z) \langle Z^g, W \rangle = \langle Z, W \rangle \iff (\forall Z) \langle Z, W^g - W \rangle$$

$$\Longleftrightarrow W = W^g.$$

It follows that the orthogonal complement to the image of the derivative of the commutator map is the intersection of $Z(g_1)$ and $Z(g_2)^{g_1^{-1}}$, which is the intersection of $Z(g_1)$ and $Z(g_2)$. \square

The assumption in 3.1 is necessary in the following sense:

Proposition 3.2. *Let K be a finite field and $\bar{g} \in G(K)$ an element such that, for any \bar{g}_1, \bar{g}_2 satisfying $\bar{g} = [\bar{g}_1, \bar{g}_2]$, the group $\langle \bar{g}_1, \bar{g}_2 \rangle$ has a fixed point in $\mathfrak{g}^*(K)$. Then there exists a local ring O with residue field K and some lift of \bar{g} to $G(O)$ which is not a commutator in $G(O)$.*

Proof. Let $O = K \oplus \bigoplus_{x \in \mathfrak{g}(K)} e_x K$, where the elements e_x satisfy $\forall x, y \in \mathfrak{g}(K), e_x e_y = 0$. We identify $G(O)$ with $G(K) \times \prod_{x \in \mathfrak{g}(K)} \mathfrak{g}(K) e_x$, and let g be the element $g = \bar{g} \times \prod_{x \in \mathfrak{g}(K)} x e_x \in G(O)$. If g is a commutator then there exist \bar{g}_1 and \bar{g}_2 such that for any x the equation

$$x = (1 - \bar{g}_2) \cdot h_1 + (1 - \bar{g}_1^{-1}) \cdot h_2$$

has a solutions, but this contradicts the assumption on \bar{g} . \square

Corollary 3.3. *There exists a local ring O such that not every element in $\mathrm{SL}_n(O)$ is a commutator.*

Proof. The element $\bar{g} = I$ satisfies the conditions of Lemma 3.2. \square

On the other hand, Proposition 3.1 implies

Corollary 3.4. *Suppose that $\lambda \in K$ is a primitive n -th root of 1, that $\bar{g} = \lambda I$, and that $g \in \mathrm{SL}_n(O)$ is some lift of \bar{g} . Then g is a commutator in $\mathrm{SL}_n(O)$.*

Proof. By Proposition 3.1, it is enough to show that there are $\bar{g}_1, \bar{g}_2 \in \mathrm{SL}_n(K)$ such that $[\bar{g}_1, \bar{g}_2] = \lambda I$ and there is no non-zero vector in $\mathfrak{sl}_n^*(K)$ that is fixed by both \bar{g}_1 and \bar{g}_2 . Let \bar{g}_1 be the diagonal matrix with diagonal $1, \lambda, \lambda^2, \dots, \lambda^{n-1}$, and let \bar{g}_2 be the permutation matrix representing the cycle $(1, 2, \dots, n)$. If $X \in \mathfrak{sl}_n^*(K)$ satisfies $\bar{g}_1 \cdot X = X$, then X is a diagonal matrix. If, in addition, $\bar{g}_2 \cdot X = X$, then X must be scalar. The existence of λ implies that n is prime to the characteristic of K , so $\mathrm{trace}(X) = 0$ implies that $X = 0$. \square

Theorem 3.5. *If O is a local ring whose residue field has more than $n + 1$ elements, then every element of $\mathrm{SL}_n(O)$ that is not scalar modulo \mathfrak{m} is a commutator.*

To show this we need the following.

Lemma 3.6. *If $A \in \mathrm{GL}_2(O)$ is not central modulo \mathfrak{m} and $\alpha \in O$, then there is $X \in \mathrm{SL}_2(O)$ such that $(XAX^{-1})_{1,1} = \alpha$.*

Proof. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ has determinant 1, then the top left entry in XAX^{-1} is $axw - bxz + cyw - dyz$. It is easy to see that there is a solution to $axw - bxz + cyw - dyz = \alpha$ and $xw - yz = 1$; For instance one can argue as follows. If $b \notin \mathfrak{m}$, taking $y = 0, x = w = 1$, there is z such that $\det X = 1$ and the top left entry of XAX^{-1} is α . If $c \notin \mathfrak{m}$ the same reasoning for A^t applies. If both b and c are in \mathfrak{m} , in view of Hensel's lemma, it is enough to show that there is a rational point of the intersections of the reductions of the two equations modulo \mathfrak{m} where the equations intersect transversely. Choose β and γ such that $a\beta - d\gamma = \alpha$ and $\beta - \gamma = 1$, and choose x, y, z, w such that $xw = \beta$ and $yz = \gamma$. \square

In the following, if A is an n -by- n matrix, we write $A = \begin{pmatrix} a & u^T \\ v & B \end{pmatrix}$, where a is a scalar, B is an $(n - 1)$ -by- $(n - 1)$ matrix, and u and v are column vectors.

Lemma 3.7. *Suppose that $n \geq 3$, that $A \in \mathrm{GL}_n(O)$ is not scalar modulo \mathfrak{m} , and that $a \in O^*$. Then there is a conjugate of A of the form $\begin{pmatrix} a & w^T \\ z & C \end{pmatrix}$, where the matrix $aC - zw^T$ is non-scalar modulo \mathfrak{m} .*

Proof. Suppose that $A = \begin{pmatrix} \alpha & u^T \\ v & B \end{pmatrix}$. Since A is non-scalar modulo \mathfrak{m} , we can conjugate it so that v is non-zero modulo \mathfrak{m} . For every $x \in O^{n-1}$,

$$\begin{pmatrix} 1 & x^T \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha & u^T \\ v & B \end{pmatrix} \begin{pmatrix} 1 & -x^T \\ 0 & I \end{pmatrix} = \begin{pmatrix} \alpha + x^T v & u^T + x^T B - \alpha x^T - (x^T v)x^T \\ v & B - vx^T \end{pmatrix},$$

and therefore we can find x such that $\alpha + x^T v = a$. Conjugating by this matrix, we can assume that $A = \begin{pmatrix} \alpha & u^T \\ v & B \end{pmatrix}$, where $\alpha = a$ (and u, v, B might have changed). Now it is enough to find x such that

1. $x^T v = 0$.
2. The matrix $\alpha B - vu^T - vx^T B$ is non-scalar modulo \mathfrak{m} .

Denote the matrix $\alpha B - vu^T$ by M . If the first condition holds, then the expression in the second condition is equal to $M - \alpha^{-1}vx^T M$.

If M is non-scalar modulo \mathfrak{m} , we can take $x = 0$. Otherwise, we can take any x such that $x^T v = 0$ since the sum of a scalar matrix and a non-zero rank-one matrix is never scalar. \square

Proof of Theorem 3.5. Suppose that $A \in \mathrm{GL}_n(O)$ is not central modulo \mathfrak{m} . Choose $a_1, \dots, a_n \in O$ such that $\prod a_i = \det(A)$. We shall prove, by induction on n , that there is a lower-triangular unipotent matrix X , an upper-triangular unipotent matrix Y , and a matrix $g \in \mathrm{SL}_n(O)$ such that $XA^gY = D$, where D is the diagonal matrix with diagonal a_1, \dots, a_n . We start with $n = 2$. By Lemma 3.6, there is $g \in \mathrm{SL}_2(O)$ such that $(A^g)_{1,1} = a_1$. There is a unipotent lower-triangular matrix X such that XA^g is upper-triangular. Similarly, there is a unipotent upper triangular matrix Y such that XA^gY is diagonal with diagonal $a_1, a_2 = \det(A)/a_1$. For the induction step, Lemma 3.7 implies that there is g such that A^g has the form $\begin{pmatrix} a_1 & u^t \\ v & B \end{pmatrix}$. Let $X = \begin{pmatrix} 1 & 0 \\ -v/a_1 & \mathrm{Id} \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -u/a_1 \\ 0 & \mathrm{Id} \end{pmatrix}$. Then $XA^gY = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$, where C is non-scalar modulo \mathfrak{m} . Applying the induction hypothesis to C , we get the claim for A .

Now assume that $A \in \mathrm{SL}_n(O)$ is not a scalar modulo \mathfrak{m} , and choose the a_i to satisfy that $\prod a_i = 1$, that $a_i = a_{n+1-i}^{-1}$ for all $i = 1, \dots, n$ and that a_i is not congruent to a_j modulo \mathfrak{m} if $i \neq j$ (this is where we use the assumption that the residue field has size greater than $n + 1$). By what we have shown, there is a lower-triangular unipotent matrix X , an upper-triangular unipotent matrix Y , and a matrix $g \in \mathrm{SL}_n(O)$ such that $X^{-1}A^gY^{-1} = D^2$, where D is the diagonal matrix with diagonal a_1, \dots, a_n . Since $A^g = (XD)(DY)$ and since XD and $(DY)^{-1}$ are conjugate (both are cyclic and have the same eigenvalues), it follows that A^g is a commutator. Hence A is also a commutator. \square

Theorem 3.8. *If O is a local ring whose residue field K elements, where n is a proper divisor of $|K| - 1$, then every element in $\mathrm{PSL}_n(O)$ is a commutator.*

Proof. By assumption, there is a primitive n -th root of 1 in K ; denote it by λ . Let $g \in \mathrm{PSL}_n(O)$. If \bar{g} is not the identity, let $h \in \mathrm{SL}_n(O)$ be any lift

of g . If $\bar{g} = 1$, let $h \in \mathrm{SL}_n(O)$ be a lift of g such that $\bar{h} = \lambda I$. By either Proposition 3.5 or Corollary 3.4, the element h is a commutator, and, hence, so is g . \square

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